Microcausality of spin-induced noncommutative theories

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Abstract

In this brief report, the microcausility of quantum field theory on spin-induced noncommutative spacetime is discussed. It is found that for spacelike seperation the microcausality is not obeyed by the theory generally. It means that Lorentz covariance can not guarantee microcausality in quantum field thoery. We also give some comments about quantum field thoeries on such noncommutative spacetime and the relations between noncommutative spacetime and causality.

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I. INTRODUCTION

One of the most important problems in theoretical physics is the quantization of gravity. There are several approaches in this direction, such as string theories and loop quantum gravity. Although we are still far from a complete theory of quantum gravity, some common notions are obtained. For example, at small scale or Planck scale, the smooth Riemann manifold structure of the large spacetime will disappear and the spacetime will manifest some quantum or discrete properties. The first discrete spacetime model was suggested by Snyder [1] at 1947 and physical models on noncommutative spacetime are studied extensively during last decades. Here we refer two reviews [2, 3]. The most common noncommutative spacetime model is the canonical model, where the spacetime operators satisfying the following relations

$$[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is constant noncommutative parameters. This model is not only a simple one but can be derived from string theory when there are some backgroud gauge fields on branes [4–7]. Using Moyal-Weyl correspondence, noncommutative field $\phi(\hat{x})$ defined on eq.(1) can be described by common field $\phi(x)$ with Moyal product [8],

$$\phi_1 * \phi_2(x) = \exp(\frac{i}{2}\theta^{\mu\nu}\partial_{\mu}^{x}\partial_{\nu}^{y})\phi_1(x)\phi_2(y)|_{y\to x}. \tag{2}$$

We encode the noncommutativity or nonlocality into the Moyal product. The above noncommutative models have one problem that it breaks Lorentz invariance explicitly. In [9, 10], the authors suggested a twisted Lorentz symmetry to resolve this problem.

Recently, an interesting Lorentz covariant noncommutative model is suggested [11, 12]. Some phenomenological applications of this model are also been studied[13].In this model, noncommutativity is related with the spin of the fields. For a scalar particle, no spacetime nocommutativity can be felt. But a Dirac particle can feel it. The commutation relations of this model for a Dirac field are

$$[\hat{x}^\mu, \hat{x}^\nu] = -i\theta\epsilon^{\mu\nu\rho\sigma}\hat{S}_{\rho\sigma} + \frac{i}{2}\theta^2\epsilon^{\mu\nu\rho\sigma}\hat{W}_\rho\hat{p}_\sigma, \tag{3}$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}. \tag{4}$$

$W_\rho = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{S}_{\rho\sigma}\hat{p}_\nu$ is the Pauli-Lubanski vector operator. And $\hat{S}_{\rho\sigma}$ are commutators between Dirac γ matrices and defined as

$$\hat{S}_{\rho\sigma} = -\frac{i}{4}(\gamma_\rho\gamma_\sigma - \gamma_\sigma\gamma_\rho) = \frac{i}{2}\gamma_{\rho\sigma}. \tag{5}$$

In this paper, we discuss the quantum microcausality of spin-noncommutative quantum field theories. The microcausality of noncommutative quantum field theories based on eq.(1) has been discussed by several authors. In[14, 15], the authors have studied microcausality by computing the commutator of operators of observables $\mathcal{O} =: \phi * \phi :$ at spacelike seperation and found that microcausality is obeyed provided that $\theta^{0i} = 0$. But it is showed that the commutators for observables with partial derivatives of fields do not vanish at spacelike seperation[16]. Similar results are obtained by other authors [17] in a different approach in noncommutative scalar and Yukawa theories. The microcausility of noncommutative scalar field theory is revisited in a more mathematical method-distributions theory in[18] and the author proved that microcausality is violated for observables $\mathcal{O}$ even for space-space noncommutativity. All the above negative results are obtained for noncommutative field theories based on eq.(1). Maybe we can see these results from a simple viewpoint-they violate common Lorentz covariance. How about a Lorentz covariant noncommutative field theory? Spin-noncommutative field theories are suitable models for consideration. In section 2, we study two examples in spin-noncommutative Dirac field theory.

II. CALCULATION OF CAUSALITY

For spin noncommutative algebras eq.(3) and eq.(4) of a massive Dirac spinor field with mass m, we have the following representation [12],

$$\hat{x}^\mu = x^\mu - \frac{i\theta}{2}\gamma^5\gamma^{\mu\nu}\partial_\nu, \qquad \hat{p}^\mu = -i\partial^\mu. \tag{6}$$

Similar to Moyal product, one can define a new star product into which we can encode the spin noncommutativiy

$$(f * g)(x) = f(x)\exp(\frac{i\theta}{2}\overleftarrow{\partial}_\mu\gamma^5\gamma^{\mu\nu}\overrightarrow{\partial}_\nu)g(x). \tag{7}$$

In the expansion of Dirac fields, we use the normalization and convention as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_p}}\sum_s(\hat{b}_{p,s}u(p,s)e^{-ip\cdot x} + \hat{d}^+_{p,s}v(p,s)e^{ip\cdot x}),$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_p}}\sum_s(\hat{b}^+_{p,s}\bar{u}(p,s)e^{ip\cdot x} + \hat{d}_{p,s}\bar{v}(p,s)e^{-ip\cdot x}).$$

where $\omega_p = \sqrt{p^2 + m^2}$ and u, v are the positive and negative energy solutions of Dirac equation. The creation and annihilation operators obey anticommutation relations

$$\{\hat{b}_{p,s}, \hat{b}^+_{q,r}\} = \delta^3(p-q)\delta_{sr}, \qquad \{\hat{d}_{p,s}, \hat{d}^+_{q,r}\} = \delta^3(p-q)\delta_{sr}.$$

To study microcausality, we choose Hermitian operator $\mathcal{O}(x) =: \bar{\psi}(x) * \psi(x) :$ as a sample observable. First, we consider the vacuum expectation value $\langle 0|[\mathcal{O}(x), \mathcal{O}(y)]|0\rangle$.

Using the definition of the star product, the observable $\mathcal{O}(x)$ can be written in explicit form

$$\begin{aligned}
\mathcal{O}(x) = &\int \frac{d^3p_1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{p_1}}}\int\frac{d^3p_2}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_{p_2}}}\sum_{s_1}\sum_{s_2}\\
&(\hat{b}^+_{p_1s_1}\hat{b}_{p_2s_2}\bar{u}(p_1,s_1)M(p_1,p_2)u(p_2,s_2)e^{i(p_1-p_2)\cdot x}\\
&+\hat{b}^+_{p_1s_1}\hat{d}^+_{p_2s_2}\bar{u}(p_1,s_1)M(p_1,-p_2)v(p_2,s_2)e^{i(p_1+p_2)\cdot x}\\
&+\hat{d}_{p_1s_1}\hat{b}_{p_2s_2}\bar{v}(p_1,s_1)M(-p_1,p_2)u(p_2,s_2)e^{-i(p_1+p_2)\cdot x}\\
&-\hat{d}^+_{p_2s_2}\hat{d}_{p_1s_1}\bar{v}(p_1,s_1)M(-p_1,-p_2)v(p_2,s_2)e^{-i(p_1-p_2)\cdot x}),
\end{aligned} \tag{8}$$

where the momentum-dependent matrix M is defined as

$$M(p,q) = \exp(\frac{i}{2}\theta\gamma^5\gamma^{\mu\nu}p_\mu q_\nu). \tag{9}$$

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From the definition, we can see that $M(p,q) = M(-p,-q)$, $M(p,-q) = M(-p,q)$. Then the vacuum expectation of the commutator of two observables are

$$\langle 0|[\mathcal{O}(x), \mathcal{O}(y)]|0\rangle = \int \frac{d^3 p_1}{(2\pi)^{3/2}} \frac{1}{2\omega_{p_1}} \int \frac{d^3 p_2}{(2\pi)^{3/2}} \frac{1}{2\omega_{p_2}}$$
$$\{\text{tr}[(\not{p}_1 + m)M(p_1, -p_2)(\not{p}_2 - m)M(-p_2, p_1)](e^{-i(p_1+p_2)(x-y)} - e^{i(p_1+p_2)(x-y)})\} \tag{10}$$

For the commutative limit, the matrix $M = 1$, the above equation becomes

$$\int \frac{d^3 p_1}{(2\pi)^{3/2}} \frac{1}{2\omega_{p_1}} \int \frac{d^3 p_2}{(2\pi)^{3/2}} \frac{1}{2\omega_{p_2}} (4p_1 \cdot p_2 - 4m^2)(e^{-i(p_1+p_2)(x-y)} - e^{i(p_1+p_2)(x-y)}). \tag{11}$$

When $x-y$ is spacelike, we can take $x_0 = y_0$ due to the Lorentz invariance of the above equation. Then it is easy to see that eq.(11) equals to zero. For noncommutative case, the difficulty in the integral is the trace term in eq.(10), which can be reduced to

$$\text{tr}[\not{p}_2 M(-p_2, p_1)\not{p}_1 M(p_1, -p_2) - m^2]. \tag{12}$$

It is hard to get an explicit expression for the above trace, but we know that the above trace is a Lorentz scalar. So it must be a function of the form $F(p_1 \cdot p_2, m^2)$. Using the same analysis of the commutative case, the eq.(10) equals to zero. We conclude that the microcausality is obeyed by vacuum expectation of the commutator.

Then let us check the matrix element between vacuum and a two-particle state. We choose the two particle state as $|q_1, r_1; q_2, r_2\rangle = \hat{b}^+_{q_1 r_1} \hat{d}^+_{q_2 r_2}|0\rangle$,

$$\langle 0|[\mathcal{O}(x), \mathcal{O}(y)]|q_1, r_1; q_2, r_2\rangle = \frac{1}{2\sqrt{\omega_{q_1}\omega_{q_2}}} e^{-iq_1 \cdot x - iq_2 \cdot y} \int \frac{d^3 p}{(2\pi)^{3/2}} \frac{1}{2\omega_p} \{(e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)})$$
$$\bar{v}(q_2, r_2)[M(-q_2, p)(\not{p} + m)M(p, q_1) - M(-q_2, -p)(\not{p} - m)M(-p, q_1)]u(q_1, r_1)\}. \tag{13}$$

At this case, we choose a definite two-particle state, so the above matrix element is not Lorentz invariant. For simplicity, the commutator is taken to be equal time. It is conceivable that the above integral is not zero and thus the microcausility is violated. But it is not easy to obtain a explicitly analytic results from it. In order to do this, we should take some limit and choose special spinor states for the two particles. In chiral representation, the matrix in the second line of the above equation

$$[M(-q_2, p)(\not{p} + m)M(p, q_1) - M(-q_2, -p)(\not{p} - m)M(-p, q_1)]$$

can be written explicitly as

$$
\begin{pmatrix}
m(e^{\vec{A}\cdot\vec{\sigma}}e^{\vec{C}\cdot\vec{\sigma}} + e^{-\vec{A}\cdot\vec{\sigma}}e^{-\vec{C}\cdot\vec{\sigma}}) & e^{\vec{A}\cdot\vec{\sigma}}(p_0 - \vec{p}\cdot\vec{\sigma})e^{\vec{D}\cdot\vec{\sigma}} - e^{-\vec{A}\cdot\vec{\sigma}}(p_0 - \mathrm{p}\cdot\vec{\sigma})e^{-\vec{D}\cdot\vec{\sigma}} \\
e^{\vec{B}\cdot\vec{\sigma}}(p_0 + \vec{p}\cdot\vec{\sigma})e^{\vec{C}\cdot\vec{\sigma}} - e^{-\vec{B}\cdot\vec{\sigma}}(p_0 + \vec{p}\cdot\vec{\sigma})e^{-\vec{C}\cdot\vec{\sigma}} & m(e^{\vec{B}\cdot\vec{\sigma}}e^{\vec{D}\cdot\vec{\sigma}} + e^{-\vec{B}\cdot\vec{\sigma}}e^{-\vec{D}\cdot\vec{\sigma}})
\end{pmatrix}
$$

where

$$
A = \frac{i\theta}{2}(p^0\vec{q}_2 - q_2^0\vec{p} + i\vec{q}_2 \times \vec{p}) \tag{14}
$$

$$
B = \frac{i\theta}{2}(p^0\vec{q}_2 - q_2^0\vec{p} - i\vec{q}_2 \times \vec{p}) \tag{15}
$$

$$
C = \frac{i\theta}{2}(p^0\vec{q}_1 - q_1^0\vec{p} + i\vec{q}_1 \times \vec{p}) \tag{16}
$$

$$
D = \frac{i\theta}{2}(p^0\vec{q}_1 - q_1^0\vec{p} - i\vec{q}_1 \times \vec{p}). \tag{17}
$$

If we take the momenta of the particles to be zero, i.e. $q_1 = q_2 = (m, 0, 0, 0)$ and the seperation to be $\vec{x} - \vec{y} = (0, 0, z)$, it turns out that the integral in eq.(13) is zero. Microcausality is preserved by this special case. Then we should consider other cases. For the sake of simplicity, we take massless limit and the momenta are chosen to be $q_1 = q_2 = (q, 0, 0, q)$ and the seperation to be $\vec{x} - \vec{y} = (0, 0, z)$. The spinor states of the two particles are $u = v \propto \begin{pmatrix} \eta \\ 0 \end{pmatrix}, \eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then the integral (up to an irrelevant coefficient) in eq.(13) is

$$
\frac{1}{4(z^2 - \theta^2 q^2)^2} \int_0^\infty d\mathrm{p}[-2i\theta qz\cos(z\mathrm{p})(e^{2i\theta q\mathrm{p}} + e^{-2i\theta q\mathrm{p}}) + 4i\theta qz\cos(z\mathrm{p})
$$
$$
-(\theta^2 q^2 + z^2)\sin(z\mathrm{p})(e^{2i\theta q\mathrm{p}} - e^{-2i\theta q\mathrm{p}}) + 4i\theta q(z^2 - \theta^2 q^2)\mathrm{p}\sin(z\mathrm{p})]
$$
$$
= \frac{\pi}{4(z^2 - \theta^2 q^2)^2}[i(z - \theta q)^2\delta(z + 2\theta q) + i(z + \theta q)^2\delta(z - 2\theta q) + 4i\theta qz\delta(z)
$$
$$
-4i\theta q(z^2 - \theta^2 q^2)\delta'(z)]. \tag{18}
$$

This matrix element is not zero. So microcausality is violated by this case. This result is similar to the one in canonical noncommutative models where the violation is also proportional to sum of δ-functions [16]. We can also see that for commutative limit, $\theta \to 0$, the integral eq.(18) is zero and microcausality is preserved as is expected. If $z \to 0$, the integral is zero too as it should be for a commutator of the same operator.

III.   CONCLUSION AND DISCUSSION

In this report, we discuss the problem of microcausality of an interesting Lorentz invariant noncommutative field theory where the noncommutativity is induced by the spin of the field. We take $\mathcal{O}(x) =: \bar{\psi}(x) * \psi(x)$: as a sample observable and show explicitly by two examples that microcausality is violated for the theory in general. In the canonical noncommutative theories with constant noncommutative parameters $\theta^{\mu\nu}$, the noncommutativity appears as extra phase factors depending on external or internal momenta. If there is only space-space noncommutativity[19], the microcausality is preserved by observables without time derivatives[16], and for general observables, microcausality is violated. In this spin-induced noncommutative model, the noncommutativity also depends on the momenta of external or internal momenta. From eq.(18), one can see that the violation of the microcausality consists of sum of δ-functions and nonlocality is proportional to the momenta of external particles. These features are similar to the ones found in canonical Moyal-type noncommutative theories [16]. In Moyal-type noncommutative theories, there are UV/IR mixing problems. One interesting thing is to explore whether there are similar problems in quantum field theories on this spin-induced noncommutative spacetime.

In classical special relativity and common quantum field theories, Lorentz covariance preserves causality and guarantees that one can not travel back through time. But on noncommutative spacetime, even Lorentz covariance can not preserve causality. It is also noted that another Lorentz-covariant noncommutative model is proposed by[20] and the Lorentz violation appears when noncommutative parameters integrated over all possible values and directions[21]. We emphasize that this nonlocality spreading procedure don't appear in spin-induced noncommutative theory. In both cases, Lorentz covariance of the action doesn't mean microcausality or local commutativity. This fact supports a viewpoint, proposed by Greenberg[22], that the Lorentz covariance of time-ordered product of fields leads to microcausality. Anyway, it is a really nontrival work to construct a quantum theory of spacetime to be consistent with causality.

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